NRL Report 7577

Solution Graphs: Simple Algebraic Structures for Problems in Linear Anisotropic Elasticity

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CONTENTS

ABSTRACT

The use of directed graphs to explicitly state the structure of solutions in plane linear anisotropic elasticity aids research into the strength and failure of composite materials by providing a visible structure for: recording available solutions, adding on new solutions, and selecting useful solutions for the problem of finding functions on the composite structure whose critical values are load and geometry invariant.

Further aid is given by recent developments in tensor manipulations in complex coordinates, the mechanics of which are included as an appendix.

The solution graphs (directed graphs) for the anisotropic half plane and for the contact of dissimilar anisotropic half planes are presented.

PROBLEM STATUS

This is a final report on one phase of a continuing NRL problem under the sponsorship of the Office of Naval Research.

AUTHORIZATION

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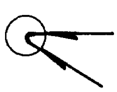
NOTATIONS

SECTION II

$T^{up}(z^n)$	complex contravariant components of the state-of-stress tensor
$E^{st}(z^n)$	contravariant complex components of the state-of-strain tensor
S^{up}_{st}	complex components of the material compliance tensor
$\vert_u = \dfrac{\partial}{\partial z^u}$	the gradient operator
S^+	the region to the left of the boundary
S^-	the region to the right of the boundary
x^u	contravariant Cartesian components of the position vector
z^u	contravariant complex components of the position vector
$T^{up}(z^n)$ (2)	node 2 of a directed graph whose value is $T^{up}(z^n)$
T^{up} E^{st} $(2) \xrightarrow{S^{st}_{up}} (7)$	directed edge of a graph which connects nodes 2 and 7 which represents a transformation S^{st}_{up}: $T^{up} \rightarrow E^{st}$
	node with edges connected internally indicates that the value of the node is to be taken as the sum of value obtained by two paths
$D^t(z^n)$	complex components of the displacement vector
$V^u(z^n)$ ◎	double circle indicates that the value of the node is to be taken as the zero of $V^u(z^n)$
$\psi(z^n)$	a zero-order tensor which provides the zero for the compatibility condition
$\perp$	the normal notation used to denote the transformation; $ip(u)T^u(z^n) = T^u(w^n) = T^{u^\perp}(z^n)$, where w^n is a coordinate system rotated 90° clockwise from z^n
$\phi(z^n)$	the real-valued Airy's stress function
$V^u(z^n)$	vector used to provide the zero for the equilibrium condition
γ_p	complex components of the tangent vector of the arc across which the stress vector acts
$\gamma_p{}^\perp$	normal vector to the arc across which the stress vector acts
$T^u(z^n)$	contravariant complex components of the stress vector
$D^{s'}(z^n)$	complex components of the derivative of the displacement vector $D^s(z^n)$
C^{up}_{st}	the complex components of the material stiffness tensor

λ^u	a characteristic vector of the polynomial differential field equation for the theory of plane anisotropic linear elasticity
${}^r\lambda^u$	the rth characteristic vector
${}^k\phi$	the kth complex-valued stress function where $\phi(z^n) = \sum_{k=1}^{4} {}^k\phi\,(z^u\, {}^k\lambda_u / \gamma^p\, {}^k\lambda_p)$
$\bar{u}$	use of the complement notation where u is an index set (1,2) and $\bar{u}$ is the index set (2,1)
$\bar{z}^u$	the complex conjugate of z^u
${}^k w$	complex arguments of the complex stress function, ${}^k\phi({}^k w)$, where ${}^k w = z^u\, {}^k\lambda_u / \gamma^p\, {}^k\lambda_p$
Γ	the $[-a,a]$ segment of the x^1 axis
${}_k\delta$	the set (1,1)
z	is z^1
$h^\ell(z)$	a discontinuous holomorphic function vector
i	$\sqrt{-1}$ (always)
t	a point of the arc Γ
t^+	t approached from the region S^+
t^-	t approached from the region S^-
δ^u_v	mixed complex components of the metric tensor and equal to the metric tensor of the Cartesian reference frame
B^s_ℓ	complex-valued tensor whose value is determined in terms of products of other tensors from the directed graph for the half plane
$X^\ell_m(z)$	a function tensor whose components are sectionally discontinuous holomorphic functions and has the property that it maps a cauchy integral operator into a space where the inverse is also a cauchy integral operator
BVI	boundary value problem of the first type, where the boundary stress vector is specified.
$\lvert B\rvert$	the determinant of B^s_ℓ

SECTION III

${}^3T^u(t)$	the discontinuity in the stress vector along the arc of contact Γ
${}^4D^{s'}(t)$	the discontinuity in the derivative of the displacement vector along the arc of contact Γ at the point t

${}^{1}T^{u}$, ${}^{1}D^{s'}$	the stress and displacement derivative vector for the body to the left
${}^{2}T^{u}$, ${}^{2}D^{s'}$	the stress and displacement derivative vectors for the body to the right
P^{g}	the resultant load vector for the body on the left and the right where ${}^{3}T^{u}(t) = 0$

APPENDIX A

$\frac{\partial}{\partial x^{u}}$	partial derivatives wrt to contravariant Cartesian components of the position vector, which transforms as a set of covariant components
g^{uv}, g_{uv}	complex components of the metric tensor
g	determinant of g_{uv}
$T^{u \ldots v}{}_{w \ldots z}(x^{n})$	Cartesian components of the general tensor, T, contravariant wrt the indices $u \ldots v$ and covariant wrt to the indices $w \ldots z$
$T^{u \ldots v}{}_{w \ldots z}(z^{n})$	complex components of the general tensor T
${}^{k}w_{\ell}(z^{n})$	set of complex components which transform as a vector wrt both k and ℓ

SOLUTION GRAPHS: SIMPLE ALGEBRAIC STRUCTURES FOR PROBLEMS IN LINEAR ANISOTROPIC ELASTICITY

I. INTRODUCTION

A literature survey of fiber-reinforced plastic composites for the years 1958 to 1970 by Beckwith et al. (1970) lists a total of 1956 publications for an average of 163 per year, whereas a similar survey for 1971 by Beckwith et al. (1971) lists 514 publications. Though the numbers are not given for the number of publications on the stress analysis of composite structures, they are probably proportional to those given above, indicating a rapidly growing use of the theory of plane linear anisotropic elasticity. For an effective use of the theory two principal difficulties need to be remedied. First, the representation of the mathematical transformations is cumbersome and at times unmanageable. Second, the mathematical structure of a problem solution is practically imperceptible.

The proposed remedy to the first difficulty is reported on in NRL Report 7537, titled "Tensor Manipulations in Complex Coordinates," which introduces into the tensor calculus for complex variable coordinate systems several new integer functions and notations which simplify greatly the form of the tensor transformations encountered. The mathematical apparatus from this report is included in Appendix A.

The proposed remedy to the second difficulty is the use of directed graphs to state explicitly the mathematical structure of the problem solutions much in the same way as diagrams in algebra and flow charts in computer programming are used.

As demonstrations of the effectiveness of these remedies, the solution graphs for the half plane and for the contact of dissimilar anisotropic half planes are constructed. The degree of effectiveness of the remedies can be judged by comparison with the work of Green and Zerna (1968) for anisotropic half planes and with the work of Clements (1971) for the contact of anisotropic half planes.

The construction of a solution graph for boundary-value problems in continuum mechanics can be broken down into three steps: First, a directed graph must be constructed, where the nodes represent the defined quantities and the edges represent the basic relations between these quantities (typified by the graph of Fig. 5 for elasticity). Second, a node must be constructed whose value implies satisfaction of the differential-constraint relationships (such as compatibility and equilibrium); then the graph can be extended so that a simple path exists from that node to the boundary-value quantities (such as the stress vector or derivative of the displacement vector at the boundary). Third, since at this stage all the node edge constructions are directed toward the boundary quantities, a node-edge sequence must be found that leads from the boundary quantities to the node mentioned above. This is done in the present case by use of a directed graph for the linear Hilbert-arc problem for function vectors. Once the graph is constructed, it becomes an entity and answers all questions about how one node can be evaluated in terms of another node of the graph.

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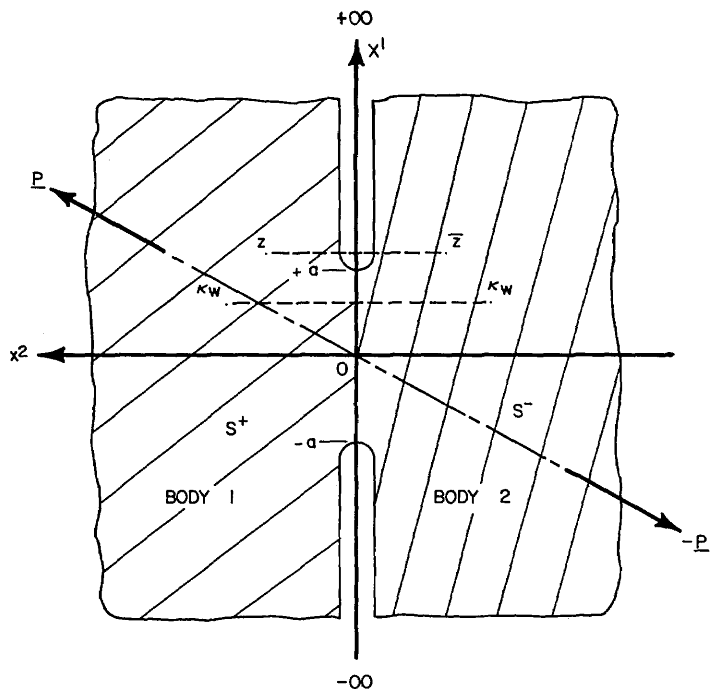


Fig. 1 — Dissimilar anisotropic bodies in contact along a segment of the real axis

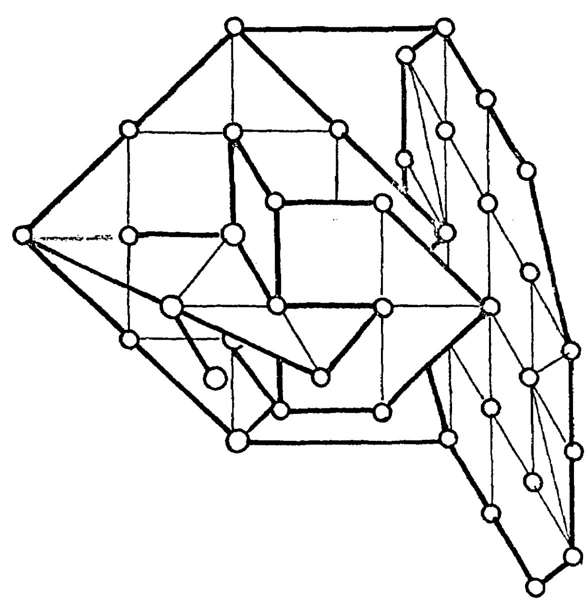


Fig. 2 — Solution graph for the linear elastic half plane S^+ in Fig. 1

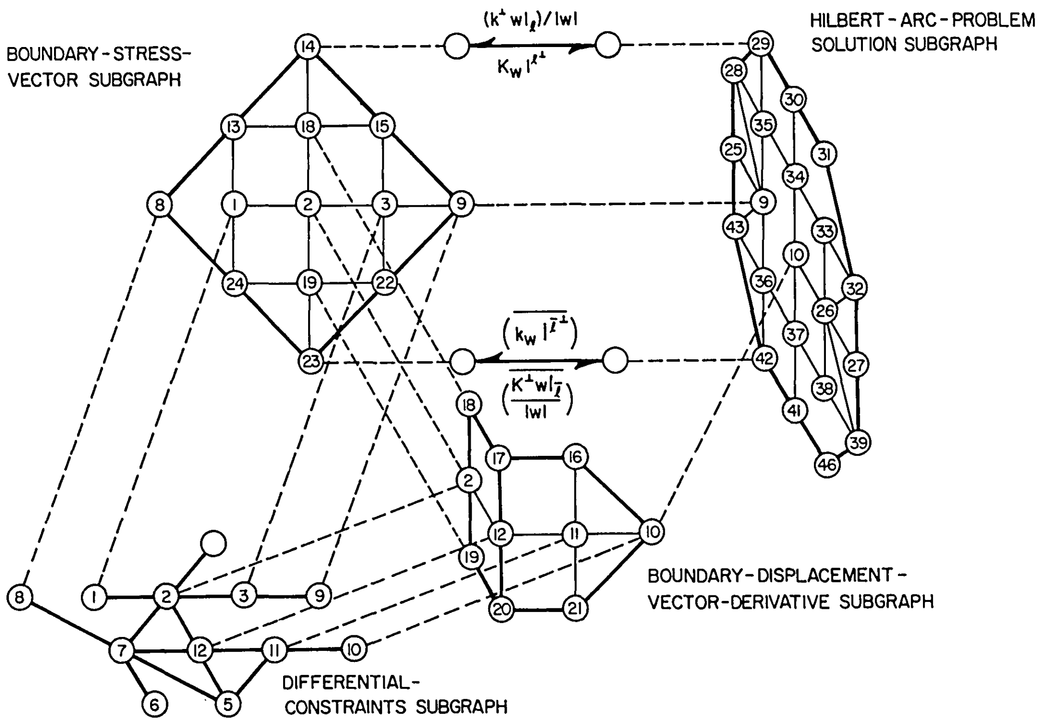


Fig. 3 — Exploded view of the solution graph shown in Fig. 2

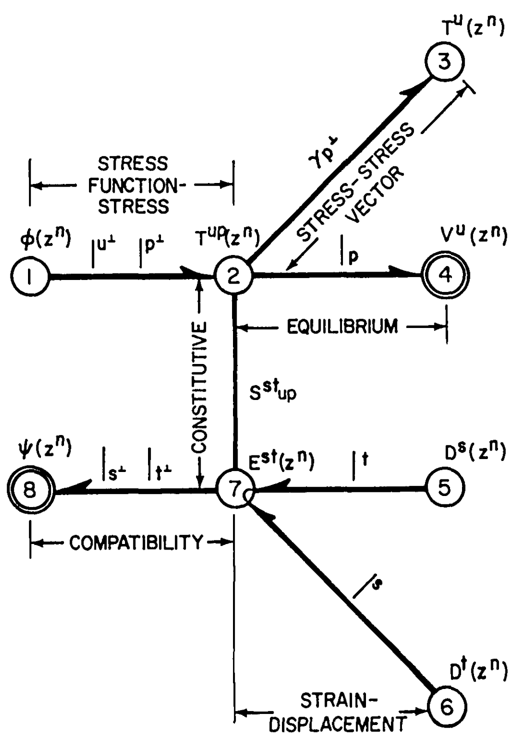


Fig. 4 — Graph of the relationship and differential constraints for plane, linear elasticity of anisotropic bodies

Differential-Constraints Subgraph

The differential relationships of plane linear elasticity are represented by the graph* of Fig. 4. The linear constitutive relationship of the state of stress $T^{up}(z^n)$ and the state of strain $E^{st}(z^n)$ is given by the sequence

$$\begin{tikzcd} \overset{T^{up}(z^n)}{\textcircled{2}} \arrow[r, "S^{st}_{up}"] & \overset{E^{st}(z^n)}{\textcircled{7}} \end{tikzcd} \tag{1}$$

and the strain displacement relations by

$$\begin{tikzcd} \overset{E^{st}(z^n)}{\textcircled{7}} & \overset{D^{s}(z^n)}{\textcircled{5}} \arrow[l, "|^{t}"'] \\ & \overset{D^{t}(z^n)}{\textcircled{6}} \arrow[ul, "|^{s}"] \end{tikzcd} \tag{2}$$

which is read as

$$\begin{aligned} E^{st} &= |^{t}D^{s} + |^{s}D^{t} \\ &= D^{s}\,|^{t} + D^{t}\,|^{s}. \end{aligned} \tag{3}$$

The compatibility condition, given by A31 (Appendix A), is represented by the sequence†

$$\begin{tikzcd} \overset{\psi(z^n)}{\textcircled{\textcircled{8}}} & \overset{E^{st}(z^n)}{\textcircled{7}} \arrow[l, "|_{s^{\perp}}\ |_{t^{\perp}}"'] \end{tikzcd} \tag{4}$$

with $\psi(z^n)$ being a zero order tensor, which provides the zero for the compatibility condition. The stress relation of Airy's stress function $\phi(z^n)$ is given by the sequence

$$\begin{tikzcd} \overset{\phi(z^n)}{\textcircled{1}} \arrow[r, "|^{u^{\perp}}\ |^{p^{\perp}}"] & \overset{T^{up}(z^n)}{\textcircled{2}} \end{tikzcd} \tag{5}$$

using the previously derived form given by A30. The equilibrium conditions are given by the sequence

$$\begin{tikzcd} \overset{T^{up}(z^n)}{\textcircled{2}} \arrow[r, "|_{p}"] & \overset{V^{u}(z^n)}{\textcircled{4}} \end{tikzcd}. \tag{6}$$

Here $V^u(z^n)$ is the first order tensor, which provides the zero vector for the relationship. The last sequence of Fig. 4 is the stress vector-stress relationship given by

$$\begin{tikzcd} \overset{T^{up}(z^n)}{\textcircled{2}} \arrow[r, "\gamma_{p^{\perp}}"] & \overset{T^{u}(z^n)}{\textcircled{3}} \end{tikzcd} \tag{7}$$

where the vector γ_p is the tangent vector to the arc across which the stress vector $T^u(z^n)$ acts and $\gamma_{p^{\perp}}$ is, by A28, the normal vector to this arc.

*Node numbering will be the same throughout the text. The half arrow indicates in which direction the transformation written on that side of the edge takes place. In addition the value of a node is to be taken as the sum of any edges which are connected within the node symbol.

†The double circle indicates that the value of the node is to be taken as zero.

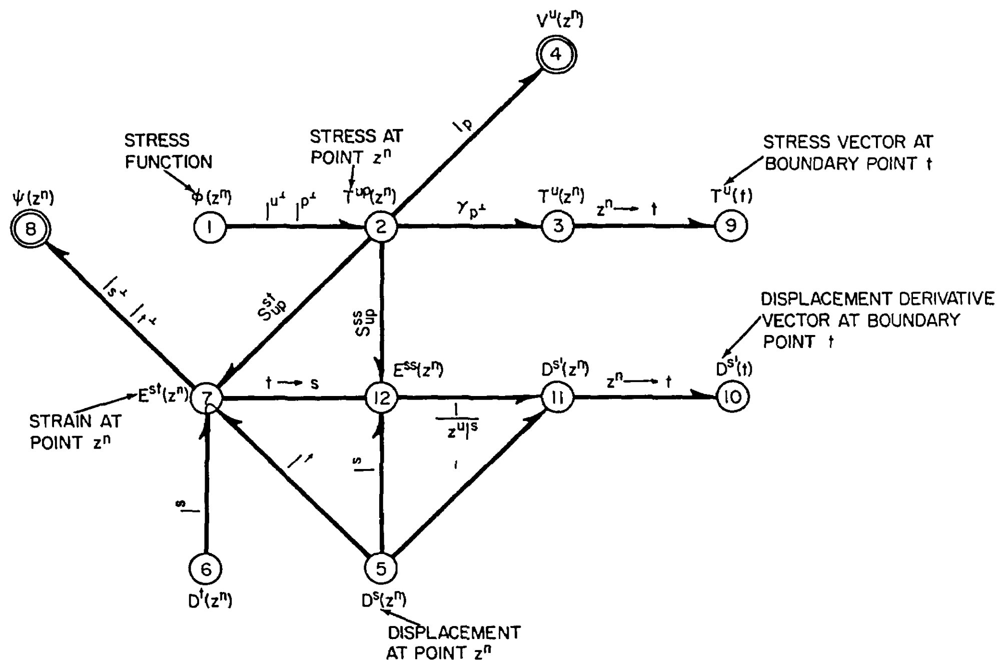
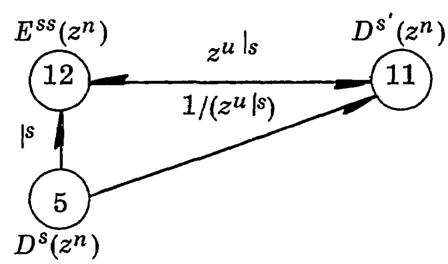


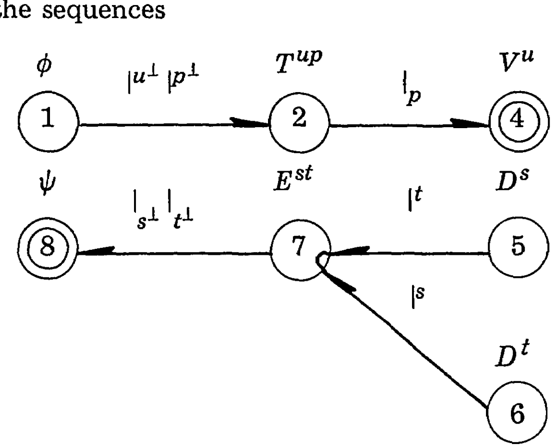
Fig. 5 — Differential-constraints subgraph of the solution graph for the half plane

Figure 5 is a rearrangement of Fig. 4 into the configuration used in the graph. In addition boundary nodes for the stress vector and the displacement derivative have been added by letting the position vector z^n approach the boundary. The node $D^{s'}(z^n)$ is obtained by using the chain rule for differentiation to obtain the sequence



(8)

The character of the sequences



(9)

should be noted. Such sequences are called exact. In an exact sequence the image of a node under the edge transformation connecting that node to the next node is the kernel of the next node, where the kernel of a node is the domain of that node which transforms to the zero element of the next node (the identity element for the node). Such sequences have been observed to occur in many different areas of mathematical analysis. The occurrence of exact sequences in the formulation of elasticity is pointed out because it does not seem to have been done so before, and use might be made of it in reorganizing the way in which the structures of continuum mechanics might be stated more clearly. For example, given the equilibrium condition, it is obvious from the notation what the stress-function definition should be in order that the stress-function definition and the equilibrium condition form an exact sequence. The property of an exact sequence that is used here is that in the sequences given by 9 any path covering the edge (1,2) insures that the edge (2,4) maps node 2 onto the zero element of node 4 and similarly any path covering edges (5,7) and (6,7) insures that node 7 maps onto the zero element of node 8.

When the sequences given by 9 are connected by the edge (2,7), two paths through the graph satisfy both the compatibility and equilibrium conditions. They are the paths (1,2,7,8) and (5,7,2,4) plus (6,7,2,4). The first gives the stress-function formulation and the second the displacement-vector formulation. The paths are represented by the differential equations

$$S^{st}_{up}\,\phi(z^n)\,|^{u^\perp p^\perp}_{s^\perp\ t^\perp} = 0, \tag{10}$$

$$C^{up}_{st}\,D^s(z^n)\,|^t_p = 0, \tag{11}$$

where C^{up}_{st} is the inverse of S^{st}_{up} and where use is made of the symmetry of C^{up}_{st}. A stress function formulation is used.

The differential equation 10, becomes, after lowering indices and noting the equivalence of the symbolization $\phi\,|_{u^\perp p^\perp} = (|_{u^\perp}\,|_{p^\perp})\phi$,

$$(S_{stup}\,|_{u^\perp}\,|_{p^\perp}\,|_{s^\perp}\,|_{t^\perp})\;\phi(z^n) = 0. \tag{12}$$

Equation 12 is a fourth-order homogeneous polynomial in $|_{u^\perp}$, which can always be factored into a product of linear forms $\lambda^u\,|_u$ (i.e. $-\,i\lambda^1(\partial/\partial z^1) + i\lambda^2(\partial/\partial z^2)$) provided λ^u is a vector in complex coordinates. When factored, 12 becomes

$$\left(\prod_{r=1}^{4}\ {}^r\lambda^u\,|_{u^\perp}\right)\,\phi(z^n) = 0, \tag{13}$$

which can be inverted by successive integration to yield

$$\phi(z^n) = \sum_{r=1}^{4}\ {}^r\phi(z^{ur}\lambda_u) \tag{14}$$

once it is noted that the integral of the differential equation

$$(\lambda^u\,|_{u^\perp})\;\phi(z^n) = 0 \tag{15}$$

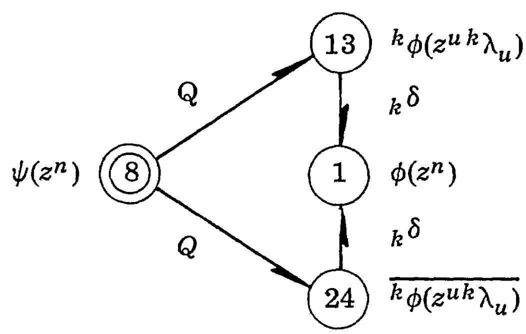
is

$$\phi(z^n) = \phi(z^u \lambda_u). \tag{16}$$

After taking the complex conjugate of 11 and 12, it is seen that if λ^u is a solution vector of 12, then $\overline{\lambda^u}$ is one also. Using this and the fact that $\phi(z^n)$ is real, equation 14 is converted to

$$\phi(z^n) = {}_k\delta\, {}^k\phi(z^u\, {}^k\lambda_u) + {}_k\delta\, \overline{{}^k\phi(z^u\, {}^k\lambda_u)}, \tag{17}$$

where $k = 1, 2$ and ${}_k\delta$ is the set (1,1). The sequences in 9 are closed symbolically by the sequence



(18)

and appears in Fig. 6.

The Continuation Coordinate System

When $z^u \to t\gamma^u$, $z^u\, {}^k\lambda_u \to t\gamma^u\, {}^k\lambda_u$ ($\|\gamma^u\| = 1$). For the purpose of a Hilbert-arc formulation of the problem the arguments of the stress functions ${}^k\phi$ must approach the same point t of the boundary. Toward this end define the variables ${}^kw$ to be

$${}^kw = \frac{z^u\, {}^k\lambda_u}{\gamma^p\, {}^k\lambda_p}, \tag{19}$$

where the boundary arc Γ is parallel to γ^p and goes through the origin. Let the region to the left of Γ be called S^+ and the region to the right be called S^-. The positive direction of Γ is that of γ^p. Since ${}^kw$ is the ratio of scalar products, it is invariant with respect to rotations and dilations of the Cartesian coordinate system.

It is easy to see that if γ^p is parallel to the real axis, $z \in S^+$, and $\bar{z} \in S^-$, then ${}^kw \in S^+$ and ${}^k\overline{w} \in S^-$. Rotating and dilating the Cartesian coordinate system obviously leaves this relationship unchanged. Changing variables $z^u\, {}^k\lambda_u / \gamma^p\, {}^k\lambda_p \to {}^kw$, 17 becomes

$$\phi(z^n) = {}_k\delta\, {}^k\phi({}^kw) + {}_k\delta\, \overline{{}^k\phi({}^kw)}. \tag{20}$$

The Boundary-Stress-Vector Subgraph

The upper half of the graph of Fig. 6 which connects ${}^k\phi({}^kw)$ to the stress vector on the boundary is constructed in several simple steps. The sequence

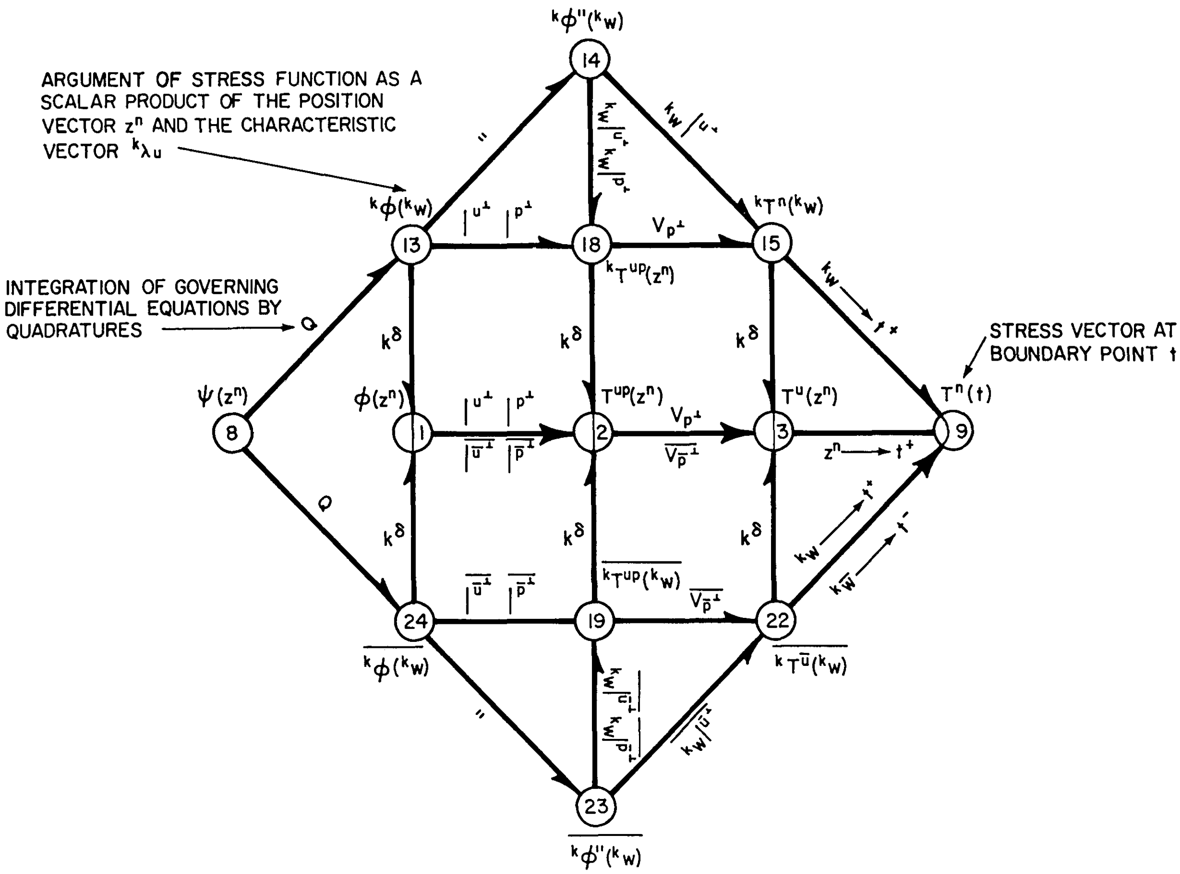


Fig. 6 — Boundary-stress-vector subgraph of the solution graph for the half plane

$$\begin{tikzcd} & (14)\ {}^k\phi''({}^k w) \arrow[d, "({}^k w\,|^{u^\perp}\ {}^k w\,|^{p^\perp})"] \\ {}^k\phi({}^k w)\ (13) \arrow[ur, "''"] \arrow[r, "|^{u^\perp}\,|^{p^\perp}"] & (18)\ {}^k T^{up}({}^k w) \end{tikzcd} \tag{21}$$

makes use of the chain rule for differentiation. Since the scalar product

$$ {}^k w\,|^{p^\perp}\ \gamma_{p^\perp} = \frac{{}^k\lambda^{p^\perp}\ \gamma_{p^\perp}}{\gamma^u\ {}^k\lambda_u} \tag{22}$$

has the value 1, the sequence

$$\begin{tikzcd} \overset{{}^k\phi''({}^k w)}{(14)} \arrow[r, "{}^k w\,|^{u^\perp}"] & \overset{{}^k T^u({}^k w)}{(15)} \end{tikzcd} \tag{23}$$

connecting ${}^k\phi''({}^k w)$ to the stress vector has a particularly simple form.

The lower half of the graph of Fig. 6 is obtained in the same manner as the upper half. Use is made of the fact that since $|^{u^\perp}$, $|^{p^\perp}$, $\gamma_{p^\perp}$ are transformable into Cartesian coordinates, by A10,

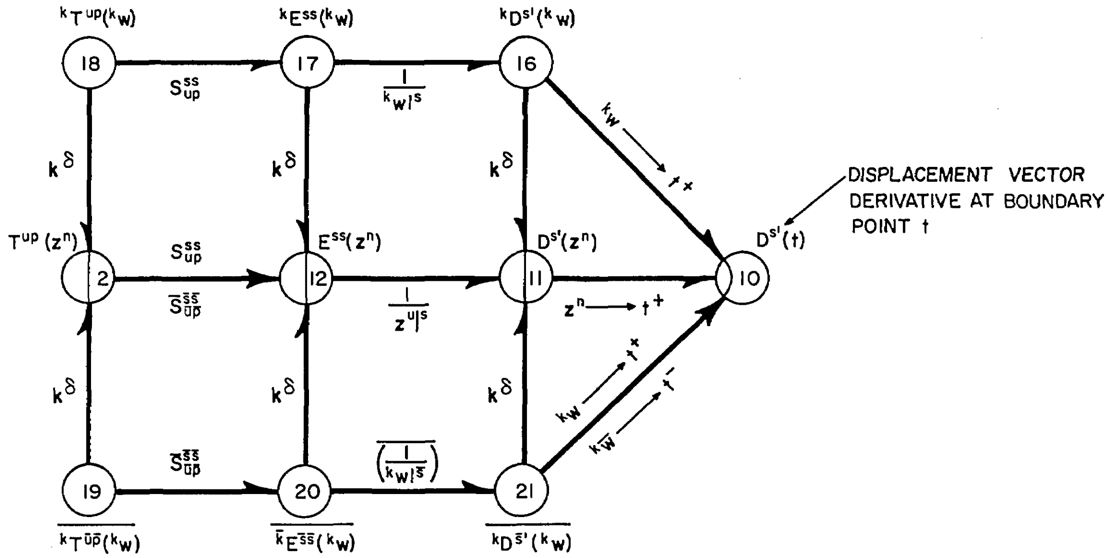


Fig. 7 — Boundary-displacement-vector-derivative subgraph of the solution graph for the half plane

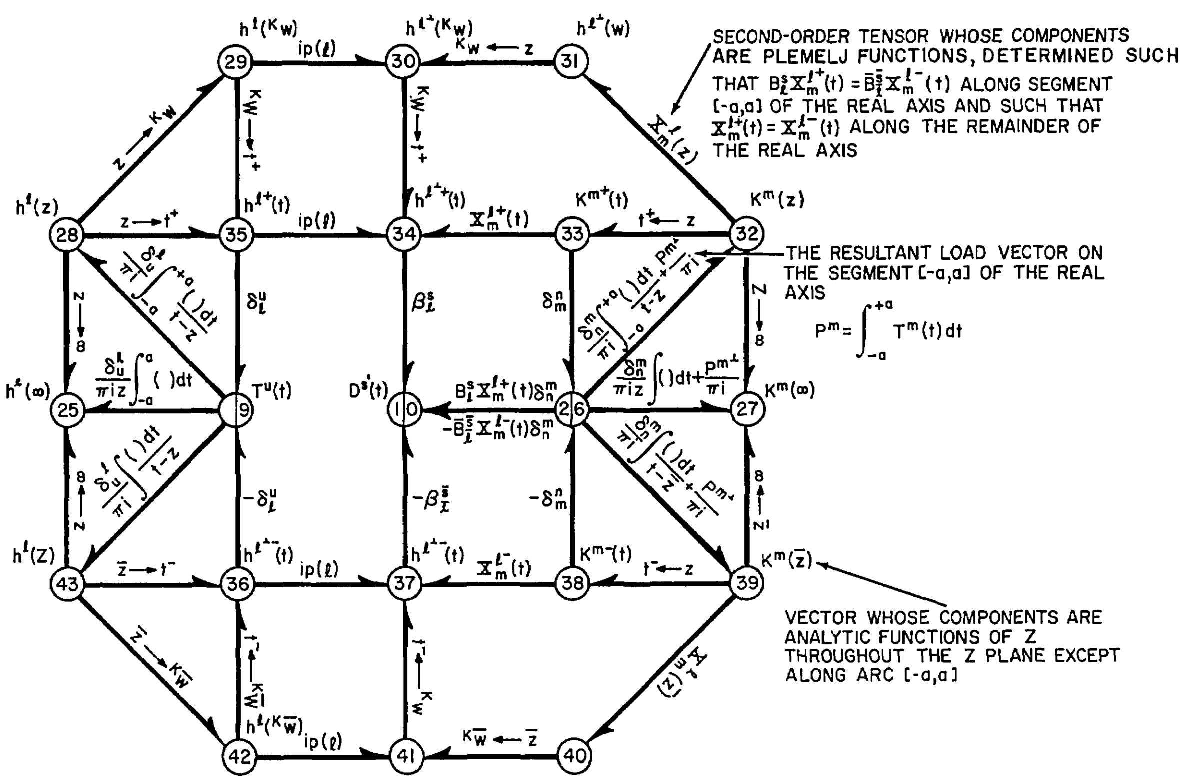


Fig. 8 — Hilbert-arc-problem solution subgraph of the solution graph for the half plane

$$|u^{\perp} = \overline{|\bar{u}^{\perp}}, \tag{24}$$

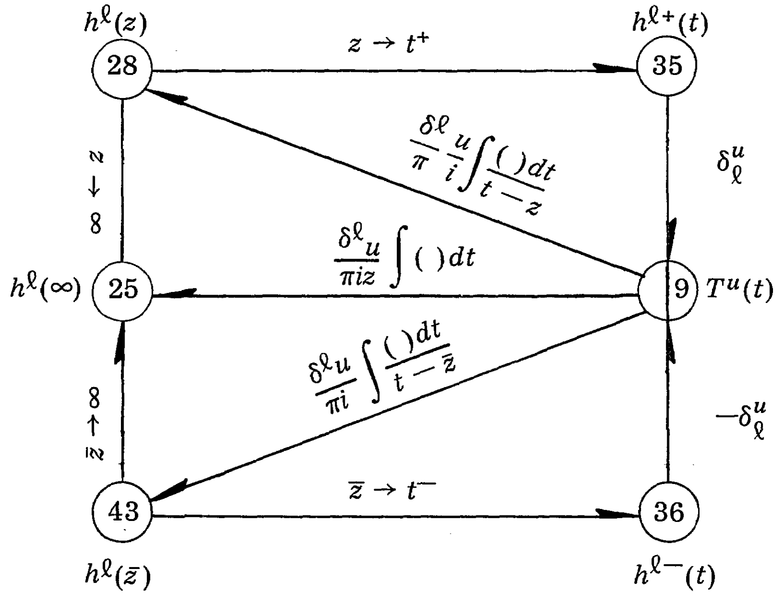
$$\gamma_{p^{\perp}} = \overline{\gamma_{\bar{p}^{\perp}}}. \tag{25}$$

The Displacement-Vector-Derivative Subgraph

The displacement derivative subgraph in Fig. 7 is constructed using the commutativity of the edge transformation to build on the subgraphs of Figs. 5 and 6.

The Hilbert-Arc-Problem Subgraph

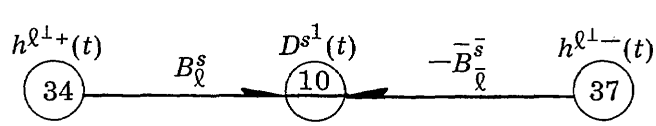
The graph of the Hilbert-arc problem is shown in Fig. 8. The graph



(26)

expresses the fact that the solution to the problem of finding a function vector $h^{\ell}(z)$ analytic throughout the z plane except along a union of arcs and curves Γ, where it satisfies the discontinuity $\delta^{u}_{\ell}\, h^{\ell+}(t) - \delta^{u}_{\ell}\, h^{\ell-}(t) = T^{u}(t)$, is found by taking the Cauchy integral of the discontinuity $T^{u}(t)$ along the arc Γ. The Cauchy integral provides the desired function vector. If poles or removable singularities are specified, the solution is modified by the addition of an appropriate polynomial.

When the problem is one of finding an analytic function vector $h^{\ell}(z)$ that satisfies the discontinuity condition



(27)

a solution can be found by transforming the sequence 27 into one similar to 26 by use of the function tensor $X^{\ell}_{m}(z)$ which satisfies the discontinuity condition

$$B^s_\ell \, X^{\ell +}_m(t) = \bar{B}^{\bar{s}}_{\bar{\ell}} \, X^{\ell -}_m(t), \; t \in \Gamma, \tag{28}$$

and is analytic elsewhere.

The transformation is shown in Fig. 8. The function vector $h^{\ell\perp}(z)$, normal to $h^\ell(z)$, is introduced in anticipation of simplifications that arise when the graph of Fig. 8 is connected to those of Figs. 5, 6, and 7.

The Half-Plane Problem

The solution graph for the half-plane problem is formed by connecting the graphs of Figs. 5, 6, 7, and 8, where the nodes (9,10) of contact have been chosen such that $h^\ell(z)$ is analytically continued through unloaded parts of the boundary.

The connections of $h^\ell(z)$ and ${}^k\phi''({}^k w)$ given by the sequences

$$\begin{tikzcd}[column sep=huge] \underset{14}{{}^k\phi''({}^k w)} \arrow[r, shift left, "{}^k w|^{\ell\perp}"] & \underset{29}{h^\ell({}^k w)} \arrow[l, shift left, "({}_{k\perp} w|_\ell)/|w|"] \\ \underset{23}{\overline{{}^k\phi''({}^k w)}} \arrow[r, shift left, "\overline{({}^k w|^{\bar{\ell}\perp})}"] & \underset{42}{h^\ell({}^k \bar{w})} \arrow[l, shift left, "\overline{({}_{k\perp} w|_{\bar{\ell}})}/\overline{|w|}"] \end{tikzcd} \tag{29}$$

are natural and correspond to the approach of Green and Zerna (1968).

For BVI problems where $T^u(t)$ is given, the value of any node can be found by evaluating the path connecting that node to node 9.

The edges (34,10) and (37,10) can now be evaluated as

$$B^s_\ell = \left(S^{ss}_{up} \, \frac{{}^k w|^{u\perp} \, {}^k w|^{p\perp} \, {}_{k\perp} w|_\ell}{|w| \, {}^k w|^s} \right), \tag{30}$$

$$\bar{B}^{\bar{s}}_{\bar{\ell}} = \left(\overline{S^{\bar{s}\bar{s}}_{\bar{u}\bar{p}} \, \frac{{}^k w|^{\bar{u}\perp} \, {}^k w|^{\bar{p}\perp} \, {}_{k\perp} w|_{\bar{\ell}}}{|w| \, {}^k w|^{\bar{s}}}} \right). \tag{31}$$

For the case of interest (where γ_p is parallel to x^1 as in Fig. 1) B^s_ℓ, when evaluated in terms of S^{st}_{up}, ${}^k\lambda_u$ becomes

$$B^1_1 = 2i \left(S^{11}_{11} - S^{11}_{22} \, \frac{{}^2\lambda_1}{{}^2\lambda_2} \, \frac{{}^1\lambda_1}{{}^1\lambda_2} \right), \tag{32}$$

$$B_2^1 = i\left(S_{12}^{11} - S_{22}^{11}\left[\frac{^2\lambda_1}{^2\lambda_2} + \frac{^1\lambda_1}{^1\lambda_2}\right]\right), \tag{33}$$

$$B_1^2 = i\left(S_{12}^{22} - S_{11}^{22}\left[\frac{^2\lambda_1}{^2\lambda_2} + \frac{^1\lambda_1}{^1\lambda_2}\right]\right), \tag{34}$$

$$B_2^2 = 2i\left(-S_{22}^{22} + S_{11}^{22}\,\frac{^2\lambda_2}{^2\lambda_1}\,\frac{^1\lambda_2}{^1\lambda_1}\right). \tag{35}$$

From this it can be established that

$$\overline{B}_{\bar{\ell}}^{\bar{s}} = |B| B_{\ell}^{s^{-1}}, \tag{36}$$

which was the result anticipated when the discontinuity condition was stated in terms of $h^{\ell\perp}(z)$ instead of $h^{\ell}(z)$. The advantage to this is that B_{ℓ}^{s} and $\overline{B}_{\bar{\ell}}^{\bar{s}}$ can be diagonalized by the same transformation. In the diagonalized form it becomes a simple matter to find Plemelj functions to satisfy condition 27. Since $X_n^{\ell}(z)$ will then be in diagonal form, B_{ℓ}^{s}, $\overline{B}_{\bar{\ell}}^{\bar{s}}$, and $X_m^{\ell}(z)$ will all commute with each other in their diagonalized or undiagonalized form. The solution of the problem where the displacement derivatives are given along part of the boundary and where the remainder is unloaded is obtained by finding the connecting path from node 10 to whatever node is desired and applying that transformation to the given values of $D^{s'}(t)$. Similar analysis for the body occupying the right half plane S^- gives the graph of Fig. 2 rotated 180° about nodes 25 and 27, with $T^u(t)$ replaced by $-T^u(t)$.

III. THE SOLUTION GRAPH FOR THE CONTACT OF DISSIMILAR ANISOTROPIC HALF PLANES AND ITS USE

An important problem for analysis is the one of contact of dissimilar anisotropic half planes over an arc $[-a, a]$ of the real axis (Fig. 1). Such an analysis is used to model such laboratory structures as a crack running at the interface of a plywood plate and an aluminum attachment, a flat crack running through a carbon filament plate, a thin variable-size failure zone propagating through a fiberglass specimen (or at an interface of an attachment of dissimilar material), or even the simple case of a crack in a homogeneous isotropic plate.

Though Muskhelishvili (1953) does not treat the case of anisotropic contact, he does treat the case of frictionless contact of dissimilar isotropic bodies. The cases were later extended by Erdogan (1963, 1965) to those of nonslipping contact over various segments of the real axis. The work of Williams (1959) and of Rice and Sih (1965) is of a similar nature but does not follow as closely that of Muskhelishvili. Clements (1971) and Willis (1971) treat the problem of contact of two dissimilar anisotropic half planes along the arcs $[-\infty, -a]$ and $[a, \infty]$ with the arc $[-a, a]$ being loaded by equal and opposite tractions on either half plane. The case of contact of dissimilar half planes along the arc $[-a, a]$ is treated here as a more useful model for a laboratory structure.

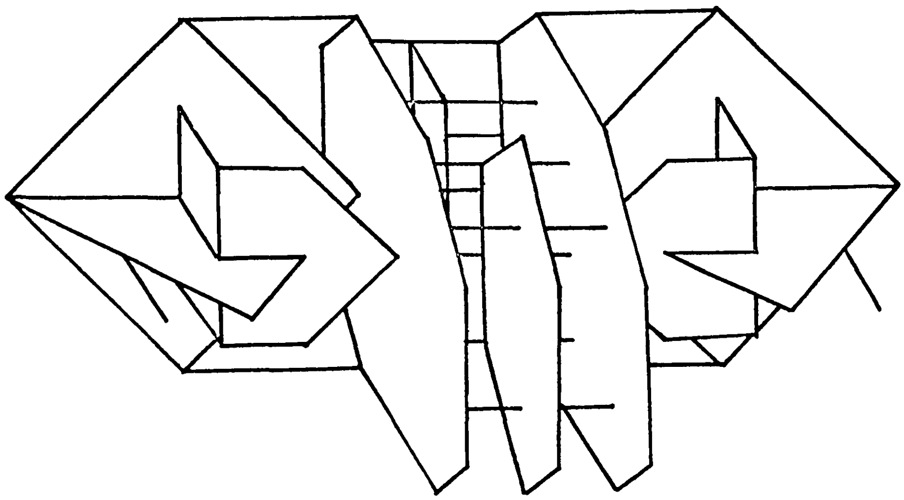


Fig. 9 — Solution graph for the linear elastic problem of the contact of dissimilar anisotropic bodies occupying the left and right half planes S^+, S^-

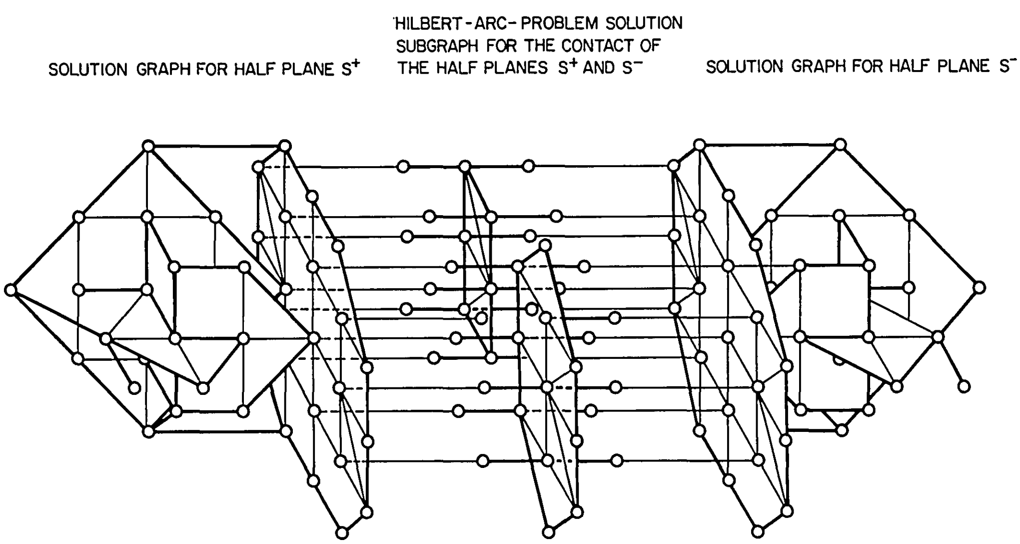


Fig. 10 — Exploded view of the solution graph in Fig. 9

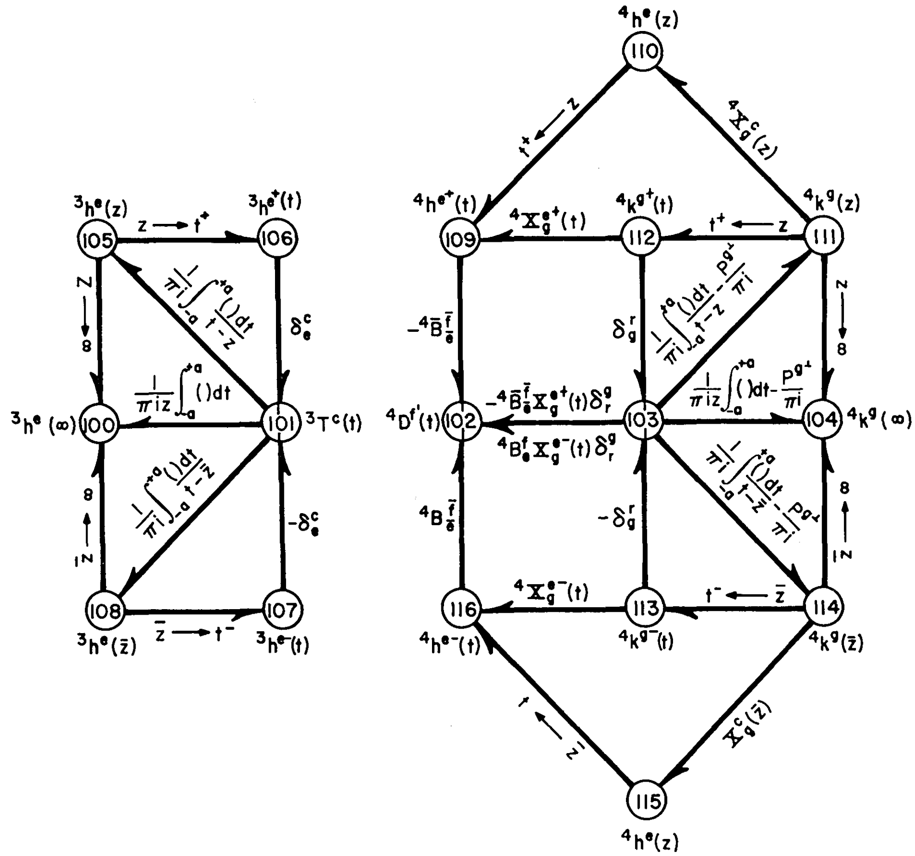
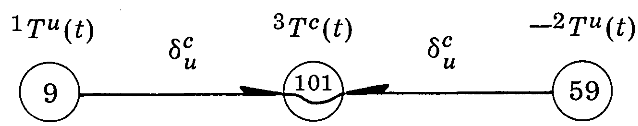


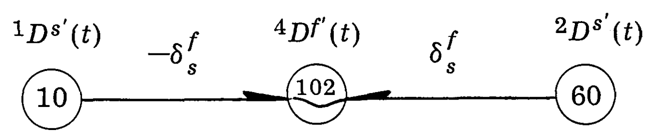
Fig. 11 — Hilbert-arc-problem subgraph of the solution graph for the contact of half planes S^+ and S^-

The solution graph for the contact problem is depicted in Fig. 9 and in exploded view in Fig. 10. The solution graph for the Hilbert-arc problem which is to be connected to the solution graphs for the right and left half planes S^- and S^+ is shown in Fig. 11 and is similar to that of Fig. 8 except that functions $h^{\ell}({}^k w)$ are not needed and that the problems for traction discontinuities ${}^3T^u(t)$ and displacement derivative discontinuities ${}^4D^{f'}(t)$ are not connected in the same graph and require the use of two separate functions, ${}^3h^{\ell}(z)$ and ${}^4h^{\ell}(z)$.

The sequences



(37)



(38)

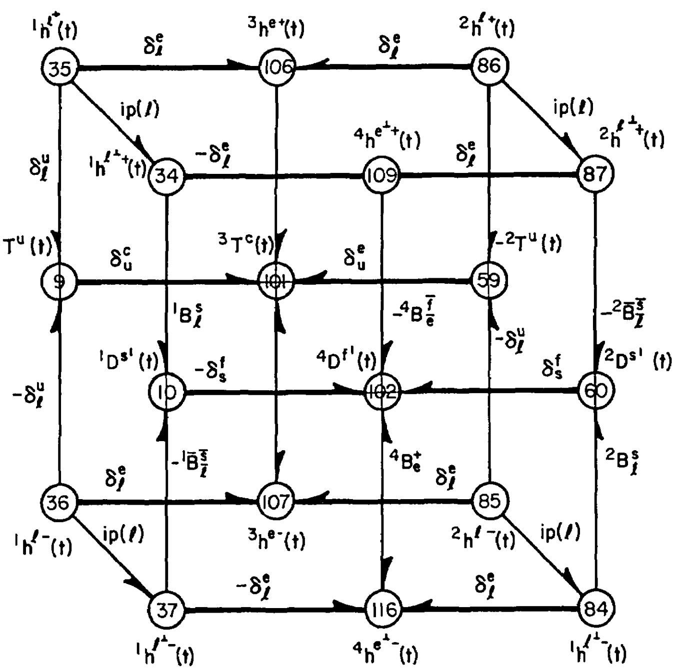


Fig. 12 — Subgraph interconnections for the contact of half planes graph

state the discontinuity conditions for $T^u(t)$ stresses and displacement derivatives across the arc $[-a, a]$. These conditions result in the desired edge connections for the half-plane problem and the Hilbert-arc problems being both positive and negative identity transformations. These are shown in Fig. 12 for a characteristic number of edges.

When the stress vector discontinuity, ${}^3T^c(t)$ is zero, the graph of Fig. 11 shows that

$$ {}^3h^e(z) = {}^1h^e(z) + {}^2h^e(z) = 0, \tag{39} $$

from which it follows that

$$ {}^1h^e(z) = -{}^2h^e(z) \tag{40} $$

and

$$ \begin{aligned} {}^4h^e(z) &= -{}^1h^e(z) + {}^2h^e(z) \\ &= -2\,{}^1h^e(z) \\ &= 2\,{}^2h^e(z). \end{aligned} \tag{41} $$

The expression for the dislocation $D^{f'}(t_0)$, where $t_0 \in [-\infty, -a] \cup [a, \infty]$ is found in terms of ${}^4D^{f'}(t)$, where $t \in [-a, a]$, from the graph of Fig. 11 as

$$ {}^4D^{f'}(t_0) = \left({}^4\overline{B}{}^{\bar{f}}_{\bar{e}} - {}^4B^{f}_{e}\right) X^{e^+}_{g}(t_0)\frac{1}{\pi i}\left[\int_{-a}^{a} \frac{\left[{}^4\overline{B}{}^{\bar{f}}_{\bar{e}} X^{e^+}_{g}(t)\right] {}^4D^{f'}(t)}{(t-t_0)}\, dt - P^{g^{\perp}}\right]. \quad (42) $$

When the dislocation ${}^4D^{f'}(t)$ is zero, 42 simplifies to

$$ {}^4D^{f'}(t_0) = \left(-{}^4\overline{B}{}^{\bar{f}}_{\bar{e}} + {}^4B^{f}_{e}\right) X^{e^+}_{g}(t)\frac{P^{g^{\perp}}}{\pi i}, \quad (43) $$

where from the graph of Fig. 12

$$ {}^4\overline{B}{}^{\bar{f}}_{\bar{e}} = {}^2\overline{B}{}^{\bar{f}}_{\bar{e}} - {}^1B^{f}_{e}, \quad (44) $$

$$ {}^4B^{f}_{e} = {}^2B^{f}_{e} - {}^1\overline{B}{}^{\bar{f}}_{\bar{e}}. \quad (45) $$

IV. SUMMARY

The effectiveness of graphing and of the extended tensor calculus in complex coordinates is demonstrated by constructing the solution graphs for the plane linear elasticity problems of anisotropic half planes and the contact of dissimilar anisotropic half planes.

In the development of the graph for the half-plane problem, the governing differential equation is interpreted as a polynomial in first-order tensors. Use of the characteristic tensors ${}^k\lambda_u$ of the polynomial result in zero-order tensor arguments of the solution functions. The edge transformations generated as derivatives of the arguments are then tensors. This is in contrast to the universal approach of using the characteristic roots, which are not tensor quantities.

The work that still remains is the extension of the graphing technique to handle any two-dimensional boundary-value problem, where solutions are sought by connection to a Hilbert-arc-problem solution graph, and the integration of this technique with symbolic manipulation computer programs (IAM, Reduce, Formac) to automate applications of the solution graph which might require tedious, lengthy, or complicated sequences of transformations.

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Appendix A

TENSOR MANIPULATIONS IN COMPLEX COORDINATES

For two-dimensional tensor analysis problems there are distinct advantages to using complex variable coordinates. One advantage is that the rotation tensors induced by rotation of the Cartesian coordinate system are diagonal (the components of tensors in the new coordinate system are scalar multiples, instead of a linear sum, of the old components). Another is that the alternating tensors take on particularly simple forms which leads to a correspondingly simplification in such things as Airy's stress function and the compatibility equation of elasticity theory. But most important a complete tensor formulation of the theory of plane linear anisotropic elasticity can be obtained in complex coordinate whereas it is not possible to obtain one in Cartesian coordinates. This follows from the fact that the roots of the characteristic equation have the transformation properties of a vector and for almost all cases are not transformable into Cartesian coordinates.

There is a potential disadvantage to the complex coordinate representation, in that a distinction is present between contravariant and covariant components, which is not present in the Cartesian case. The introduction of a new notation relieves the difficulty.

Manipulation problems are further relieved for transformations in general by introducing two integer functions.

For convenience Green and Zerna (1968) will be followed as closely as possible with respect to notation, symbolization, and development. Wherever possible a particular equation from Green and Zerna (1968) will be used as a starting point for the development of relationships.

THE TRANSFORMATION TO COMPLEX COORDINATES

Let x^u be the components of a vector in a Cartesian coordinate system and z^u the contravariant components of the same vector in a complex variable coordinate system defined by the transformation*

$$\begin{Bmatrix} z^1 \\ z^2 \end{Bmatrix} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{Bmatrix} x^1 \\ x^2 \end{Bmatrix} \frac{1}{\sqrt{2}} \tag{A1a}$$

and the inverse transformation

$$\begin{Bmatrix} x^1 \\ x^2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{Bmatrix} z^1 \\ z^2 \end{Bmatrix} \frac{1}{\sqrt{2}}, \tag{A1b}$$

where i is always to mean $\sqrt{-1}$.

*The use of $1/\sqrt{2}$ in A1a instead of the usual 1 is credited to R.J. Sanford of NRL and results in A3a not being multiplied by 2 as would have been the case.

The derivatives of the transformations are*

$$\left\{\frac{\partial z^u}{\partial x^v}\right\} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \frac{1}{\sqrt{2}} \tag{A2a}$$

and

$$\left\{\frac{\partial x^u}{\partial z^v}\right\} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \frac{1}{\sqrt{2}}. \tag{A2b}$$

The metric tensors and their determinants then follow as†

$$\left\{g^{uv}\right\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{A3a}$$

$$\left\{g_{uv}\right\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{A3b}$$

$$\det\left\{g^{uv}\right\} = (-i)^2 = -1 = 1/g, \tag{A4a}$$

$$\det\left\{g_{uv}\right\} = (i)^2 = -1 = g. \tag{A4b}$$

The metric derived from the Cartesian system by A2a is extended to those tensors in the complex system not transformable by A2b into the Cartesian system. An apparent disadvantage of this is that vectors of nonreal length can occur. A clear advantage is that it provides a means of distinguishing those tensors not transformable to Cartesian coordinates.

ASSOCIATED TENSORS AND THE COMPLEMENT NOTATION

The associated tensors‡ defined by the relationship¶

$$T^u{}_{w\ldots z} = g^{uv}T_{vw\ldots z} \tag{A5a}$$

when using A3a become

$$\begin{Bmatrix} T^1{}_{w\ldots z} \\ T^2{}_{w\ldots z} \end{Bmatrix} = \begin{Bmatrix} T_{2w\ldots z} \\ T_{1w\ldots z} \end{Bmatrix}. \tag{A5b}$$

The complement notation is defined as follows. If u is the set (1,2), then $\bar{u}$ is defined to be the set (2,1). Using this notation, A5b can now be written as

*See Section 6.4 of Green and Zerna (1968).
†See Section 1.9 of Green and Zerna (1968).
‡See Section 1.10 of Green and Zerna (1968).
¶If an index appears one or more times on only one side of an equation, then a summation over the range of the index is implied.

$$T^{u}{}_{w\ldots z} = T_{\bar{u}w\ldots z}. \tag{A6}$$

Similarly for lowering an indice the relationship is

$$T_{u}{}^{w\ldots z} = T^{\bar{u}w\ldots z}. \tag{A7}$$

TENSORS NOT TRANSFORMABLE TO CARTESIAN COORDINATES

The associated tensors and the complement notation provide a ready means for detecting which tensors are transformable to Cartesian coordinates (that is, finding the conditions on the complex components of a tensor such that under the transformation A1b the components become real). When the x^u are real, the relations

$$\begin{aligned} \bar{z} &= \frac{1}{\sqrt{2}}(x^1 - ix^2) = z^2, \\ z &= \frac{1}{\sqrt{2}}(x^1 + ix^2) = z^1, \end{aligned} \tag{A8}$$

where $\bar{z}$ means the complex conjugate of z, are true. Using the complement notation, A8 becomes

$$\bar{z}^u = z^{\bar{u}}. \tag{A9}$$

For a general tensor $T^{s\ldots t}{}_{k\ldots \ell}$ the statement corresponding to A9 is that

$$T^{s\ldots t}{}_{k\ldots \ell} = \overline{T}^{\bar{s}\ldots \bar{t}}{}_{\bar{k}\ldots \bar{\ell}} \tag{A10}$$

is true when $T^{s\ldots t}{}_{k\ldots \ell}(z^n)$ is transformable to Cartesian coordinates. This is easily shown by the following argument. Under the transformation A1b we have that

$$T^{u\ldots v}{}_{w\ldots z}(x^n) = \frac{\partial x^u}{\partial z^s} \cdots \frac{\partial x^v}{\partial z^t} \frac{\partial z^k}{\partial x^w} \cdots \frac{\partial z^\ell}{\partial x^z} T^{s\ldots t}{}_{k\ldots \ell}(z^n). \tag{A11}$$

When the x^n are real, it follows from A9 that

$$\overline{\frac{\partial x^u}{\partial z^v}} = \frac{\partial x^u}{\partial z^{\bar{v}}}, \tag{A12}$$

so that the complex conjugate of A11 is

$$\overline{T}^{u\ldots v}{}_{w\ldots z}(x^n) = \frac{\partial x^u}{\partial z^{\bar{s}}} \cdots \frac{\partial x^v}{\partial z^{\bar{t}}} \frac{\partial z^{\bar{k}}}{\partial x^w} \cdots \frac{\partial z^{\bar{\ell}}}{\partial x^z} \overline{T}^{s\ldots t}{}_{k\ldots \ell}(z^n). \tag{A13}$$

When the tensor components are Cartesian and hence real, the left sides of A11 and A13 are equal. Substituting $\bar{s} \to s$, $\bar{t} \to t$, $\bar{k} \to k$, $\bar{\ell} \to \ell$ into A13 and equating the right sides of A11 and A13, we obtain A10.

THE ALTERNATING TENSORS AND THE p FUNCTION

The Airy's stress function $\phi(z^n)$ for zero body forces is defined in terms of the stress tensor $T^{up}(z^n)$ as*

$$T^{up} = \epsilon^{us}\epsilon^{pt}\ \phi|_{st}. \tag{A14}$$

The alternating tensor $\in^{rs}$ is obtained from the tensor e^{rs} of the Cartesian system by the transformation†

$$\epsilon^{rs} = \frac{1}{\sqrt{g}} e^{rs} = ie^{rs}. \tag{A15}$$

Since r and s must be either 1 or 2, and r cannot be equal to s, an obvious simplification of A15 is obtained using the complement notation, so that A15 becomes

$$\epsilon^{rs} = \epsilon^{r\bar{r}}\,\delta^{s}_{\bar{r}} = ie^{r\bar{r}}\,\delta^{s}_{\bar{r}}. \tag{A16}$$

Since $e^{r\bar{r}}$ is +1 or —1 depending on whether $r = 1$ or $r = 2$, A16 can be further simplified by introducing the integer function $p(r)$ defined as

$$\begin{aligned} p(1) &= +1, \\ p(2) &= -1. \end{aligned} \tag{A17}$$

Using A4b and A17, A16 then becomes

$$\epsilon^{rs} = -\,ip(r)\delta^{s}_{\bar{r}}. \tag{A18}$$

The Airy's stress function now becomes

$$T^{up} = -\,p(u)p(p)\phi|_{\bar{u}\bar{p}}. \tag{A19}$$

The p function has a further use in simplifying the compatibility equation‡

$$\epsilon^{su}\epsilon^{tv}E_{st}|_{uv} = 0. \tag{A20}$$

In terms of the p function and complement notation, A20 becomes

$$-\,p(s)p(t)E_{st}|_{\bar{s}\bar{t}} = 0. \tag{A21}$$

Further simplifications are possible when the rotation transformation are introduced next.

*See Section 7.5 of Green and Zerna (1968).
†See equation 1.7.14 of Green and Zerna (1968).
‡See Pearson (1959), equation IV-33.

ROTATIONS AND THE s FUNCTION

When the Cartesian coordinates are rotated through an angle θ, the derivative of the transformation is given by*

$$\left\{\frac{\partial y^u}{\partial x^v}\right\} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}, \tag{A22}$$

and the corresponding derivative in complex coordinates is

$$\frac{\partial w^u}{\partial z^v} = \frac{\partial z^u}{\partial x^s}\frac{\partial x^t}{\partial z^v}\frac{\partial y^s}{\partial x^t}. \tag{A23}$$

Using A2a, A2b, and the p function, this becomes

$$\left\{\frac{\partial w^u}{\partial z^r}\right\} = \begin{bmatrix}e^{-i\theta} & 0 \\ 0 & e^{i\theta}\end{bmatrix} = e^{-ip(v)\theta}\delta^u_v. \tag{A24}$$

The general tensor $T^{u...v}{}_{w...z}(z^n)$ becomes under the transformation A24

$$\begin{aligned} T^{u...v}{}_{w...z}(w^n) &= \frac{\partial w^u}{\partial z^s} \cdots \frac{\partial w^v}{\partial z^t}\frac{\partial z^k}{\partial w^w} \cdots \frac{\partial z^\ell}{\partial w^z} T^{s...t}{}_{k...\ell}(z^n) \\ &= e^{i(p(\bar{u}) + ... + p(\bar{v}) + p(w) + ... + p(z))\theta} T^{u...v}{}_{w...z}(z^n). \end{aligned} \tag{A25}$$

When the integer function s is defined by

$$s(u...v) = p(u) + ... + p(v), \tag{A26}$$

then A25 becomes

$$T^{u...v}{}_{w...z}(w^n) = e^{is(\bar{u}...\bar{v}w...z)\theta} T^{u...v}{}_{w...z}(z^n). \tag{A27}$$

Of particular interest is the case of $\theta = -\pi/2$, which corresponds to rotating the coordinate system clockwise through 90° or rotating the tensors counterclockwise by 90°. For this case A27 becomes, using the fact that $e^{-ip(u)\pi/2} = ip(\bar{u})$,

$$\begin{aligned} T^{u...v}{}_{w...\ell}(w^n) &= i^n p(u)...p(v)p(\bar{w})...p(\bar{\ell})T^{u...v}{}_{w...\ell}(z^n) \\ &= T^{u^\perp...v^\perp}{}_{w^\perp...\ell^\perp}(z^n) \end{aligned} \tag{A28}$$

where the symbol $\perp$ is used to indicate the transformation

$$T^{u^\perp...v} = ip(u)T^{u...v}. \tag{A29}$$

*Covariant differentiation wrt contravariant coordinates produces covariant components, so that the tensor $\partial y^u/\partial x^v$ is controvariant wrt to u and covariant wrt to v.

Taking up again the Airy's stress function definition A19, we see that, using A29 and associated tensors, it can be written as

$$T^{up} = \phi|^{u^{\perp}p^{\perp}}. \tag{A30}$$

Similarly the compatibility equation A22 becomes

$$E_{st}|^{s^{\perp}t^{\perp}} = 0. \tag{A31}$$

THE INVERSE OF A TWO-BY-TWO MATRIX

It is convenient to have a simple expression for the inverse of a two-by-two matrix. Consider the matrix

$$\left\{{}^{k}w_{\ell}\right\} = \begin{bmatrix} {}^{1}w_{1} & {}^{1}w_{2} \\ {}^{2}w_{1} & {}^{2}w_{2} \end{bmatrix}, \tag{A32}$$

which has the inverse

$$\left\{{}^{k}w_{\ell}^{-1}\right\} = \begin{bmatrix} {}^{2}w_{2} & -{}^{1}w_{2} \\ -{}^{2}w_{1} & {}^{1}w_{1} \end{bmatrix} \frac{1}{|w|}. \tag{A33}$$

Using the p function and the complement notation, A33 becomes

$${}^{k}w_{\ell}^{-1} = p(\bar{k})p(\bar{\ell})^{\bar{k}}w_{\bar{\ell}}/|w|, \tag{A34}$$

which upon application of $\perp$ notation and the use of associated tensors, becomes

$${}^{k}w_{\ell}^{-1} = {}_{k^{\perp}}w^{\ell^{\perp}}/|w|. \tag{A35}$$

The use of indices to the left and right of the tensor symbol is to indicate that ${}^{k}w_{\ell}$ transforms as a vector with respect to both k and ℓ.

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13. ABSTRACT

The use of directed graphs to explicitly state the structure of solutions in plane linear anisotropic elasticity aids research into the strength and failure of composite materials by providing a visible structure for: recording available solutions, adding on new solutions, and selecting useful solutions for the problem of finding functions on the composite structure whose critical values are load and geometry invariant.

Further aid is given by recent developments in tensor manipulations in complex coordinates, the mechanics of which are included as an appendix.

The solution graphs (directed graphs) for the anisotropic half plane and for the contact of dissimilar anisotropic half planes are presented.

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